Optimal Linear Regulation with Hard Constraints

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An l_∞ approach to the design of linear multivariable controllers for discrete-time systems with hard time-domain constraints is presented. The notion of polar of the set of the exogenous inputs is used to parameterize the set of closed-loop transfer functions that meet regulation constraints. The constraints may include magnitude and rate bounds on all relevant process variables, including the control inputs. Solutions for optimal l_∞ design are found by solving a linear program for the impulse-response coefficients of the controller, or for the coefficients of an ARMA controller model. Using these formulations, an analytical framework is provided for delineating the tradeoffs that govern design of linear control systems.

Introduction

Performance goals for multivariable process control systems are often difficult to specify and are frequently in conflict. One aspect of performance specification is describing the type of signals that make up the inputs to the process, and the type of signals that would constitute acceptable behavior for key process variables. If the class of expected disturbances is drawn too large, then the resulting control design may be overly conservative. If, on the other hand, the disturbance class is too narrow, and therefore not indicative of the disturbances that will be encountered in the application, then the controller may perform poorly.

Much of the recent theoretical development in multivariable control has focused on the problem of computing control systems that perform well for entire classes of system inputs (Morari and Zafiriou, 1989; Boyd and Barrat, 1991). For example, the H_2 and H_∞ formulations of linear multivariable control are a result of using weighted quadratic norms to specify classes of system inputs and acceptable output behavior. These approaches generalize the classic frequency-domain technique that appears in most process-control textbooks. However, there are several difficulties with the quadratic approach, particularly the problem of incorporating time-domain constraints on signals in a nonconservative manner (Boyd and Doyle, 1987; Boyd and Barrat, 1991).

In many applications, unmeasured disturbances and noise are persistent, that is, they act continuously on the system as long as it is in operation. Such inputs have infinite energy, and therefore cannot be modeled as the bounded-energy signals required by frequency-domain approaches. Moreover, process signals are commonly described with bounds on the magnitude and rate of change. These attributes are difficult to express precisely in the frequency domain or with weighted quadratic norms.

The l_∞ norm is an obvious alternative for describing discrete-time signals where there are constraints on the instantaneous values. For a scalar signal with values given by $u = \{u(k)\}_{k=0}^{\infty} = \{u(0), u(1), u(2), \ldots\}$, the l_∞ norm is given by

$$\|u\|_\infty = \sup_{k \geq 0} |u(k)|.$$

By introducing appropriate weights, this norm can measure signal rate of change and other linear correlations. Constraints on manipulated variables often involve the magnitude and the rate-of-change; these translate directly into specifications on a weighted l_∞ norm. Output specifications also can be given in time domain; bounds may be imposed on the magnitude of the closed-loop response, its rate of change, and possible correlations among the regulated variables.

The infinity norm, when used to measure the inputs and outputs of a controlled linear system, naturally induces an operator norm. In this case, the operator norm turns out to be the l_1 norm of the impulse response sequence of the closed-loop transfer function. Minimizing the operator norm yields the l_1 optimal control problem as originally formulated by Vidyasagar (1986) for continuous time systems, and by Dahleh and Pearson (1987, 1988) for discrete-time systems.

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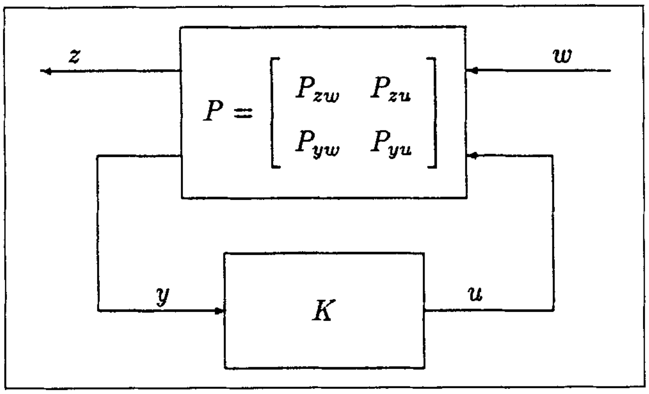

Figure 1. General feedback control system.

An important observation is that the weighted l_1 approach breaks down when the input set is defined by many l_∞ constraints. This is because the equivalent vector norm has a weighting matrix that is not full row rank; thus it cannot be inverted to form the input weighting for the closed-loop transfer function as required in the l_1 approach. As we demonstrate in this article, this restriction is significant even for the simple case of scalar systems with simultaneous magnitude and rate bounds.

The formulation proposed in this article offers a flexible approach to practical control design for systems with signals defined by weighted l_∞ norms. The main technical result is that the closed-loop regulator can be parameterized directly in terms of the disturbance set and the set of acceptable regulated responses. Using the polar of an l_∞ bounded set of signals leads to solution of the control design problem as a linear programming problem. The optimization variables are the impulse-response coefficients of the controller and some additional convex multipliers. In the case of scalar controllers, the number of parameters can be greatly reduced by using an ARMA-type controller parameterization. We show that this technique can be extended to account for a variety of performance constraints through the augmentation of the linear program.

This article is divided into three parts. In the first part we discuss time-domain specifications for the inputs and outputs of feedback systems. In the second, the nominal control problem is presented for persistent bounded signals and its solution as a linear programming problem. The third part studies a regulation problem with input constraints and model uncertainties. Examples are presented to illustrate implementation of the proposed design techniques.

Input–Output Specification in l_∞

We consider the general feedback scheme shown in Figure 1. The plant is assumed to be linear, shift invariant, and causal. In order to deal with possible constraints on the manipulated variables, the regulated output vector is normally augmented to include the control input. The input–output description of the feedback system is

$$z = P_{zu}u + P_{zw}w \tag{1}$$

$$y = P_{yu}u + P_{yw}w, \tag{2}$$

where $w \in l_\infty^{n_w}$ is a vector of exogenous inputs, $u \in l_\infty^{n_u}$ contains the manipulated variables, $y \in l_\infty^{n_y}$ contains the measured outputs, and $z \in l_\infty^{n_z}$ contains the regulated process outputs. The various P's denote linear, time invariant operators that describe the input–output behavior of the plant.

We use weighted norms to measure and place specifications on the input–output characteristics of the closed-loop system, and therefore need to introduce some terminology. The notation $x \in l_\infty^n$ refers to a semi-infinite vector sequence $\{x(0), x(1), x(2), \ldots\}$, where each $x(k)$ is in $\Re^n$ with bounded norm

$$\|x\|_\infty = \sup_{k \geq 0} \|x(k)\|_\infty \tag{3}$$

$$= \sup_{k \geq 0} \max_{1 \leq i \leq n} |x_i(k)| \tag{4}$$

$$< \infty. \tag{5}$$

Similarly, l_1 denotes the space of absolutely summable real sequences where $x \in l_1$ implies $x = \{x(k)\}_0^\infty$ with the norm $\|x\|_1 = \sum_{k=0}^\infty |x(k)| < \infty$. The set $l_1^{n \times m}$ denotes the space of matrices with individual entries belonging to l_1.

The theoretical development of this article employs conventional interpretations of discrete-time signals and operators, such as the z transform and the backward shift operator q^{-1}. For the purposes of numerical calculation, there are two practical alternatives for representing a signal $x \in l_\infty^n$. Let $\bar{x}$ denote a numerical representation of x. The elements of $\bar{x}$ can be organized as either

$$\bar{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} \\ \begin{pmatrix} x_1(1) \\ \vdots \\ x_n(1) \end{pmatrix} \\ \vdots \end{pmatrix} \tag{6}$$

or

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x_1(0) \\ x_1(1) \\ \vdots \end{pmatrix} \\ \vdots \\ \begin{pmatrix} x_n(0) \\ x_n(1) \\ \vdots \end{pmatrix} \end{pmatrix} \tag{7}$$

where $x_i(k)$ denotes the value of signal component i at time k. Each of these representations has utility. Of course, in practical calculations it will be necessary to work with finite approximations to x. In the sequel, we will not use the explicit notation $\bar{x}$, expecting that the necessary distinctions will be clear from the context.

The system models can also be represented in different ways. Consider a linear, shift-invariant operation H mapping

l_∞^n to l_∞^m. One representation of H would be as an $m \times n$ rectangular array where each element H_{ij} models a stable discrete-time transfer function. As an alternative representation for stable system, consider a numerically oriented representation where $h_{ij} = \{h_{ij}(0), h_{ij}(1), \ldots\} \in l_1$ is pulse response between output i and input j, and H_{ij} is the usual lower triangular Toeplitz matrix constructed from h_{ij}. Interpreting y and u in the sense of Eq. 7, the model $y = Hu$ can be represented as matrix multiplication where H is the block matrix with components H_{ij}. These alternative representations are used in this article, and which is in effect should be clear from the context.

The infinity norm on a signal induces an operator norm. Let $\mathcal{L}_{TI}^{n\times m}$ denote the set of linear bounded shift-invariant causal operators mapping l_∞^n to l_∞^m. A definition and a well-known formula for the induced operator norm on $\mathcal{L}_{TI}^{n\times m}$ are given by

$$\|H\|_{i,\infty} = \sup_{u \neq 0} \frac{\|Hu\|_\infty}{\|u\|_\infty} \tag{8}$$

$$= \max_i \sum_{j=1}^{n} \|h_{ij}\|_1 \tag{9}$$

$$= \max_i \sum_{j=1}^{n} \sum_{k=0}^{\infty} |h_{ij}(k)|, \tag{10}$$

where $\|h_{ij}\|_1$ is the usual norm in l_1 of the pulse response sequence h_{ij}.

Specification of input sets

All external signals that enter the system are referred to as inputs. These may include measured and unmeasured disturbances, reference inputs, and possible noise sources. These quantities are assembled to form the vector w that appears in Figure 1.

In process applications, the unmeasured disturbances can be characterized by simple bounds on magnitude, rate of change, and other linear time-domain correlations. Letting d denote unmeasured disturbances that appear as components of w, the *disturbance set* is given by

$$\mathfrak{D}_d = \{d \in l_\infty^{n_d} : \|W^d d\|_\infty \leq 1\} \tag{11}$$

where W^d is a shift-invariant stable causal weighting operator.

For example, a set of scalar signals with a magnitude bound c_m given by

$$|d(k)| \leq c_m, \quad \forall k = 1, 2, \ldots$$

is contained in the set

$$\left\| \frac{1}{c_m} d \right\|_\infty \leq 1, \tag{12}$$

where the weighting operator is the scalar $W^d = 1/c_m$.

A bound on the rate-of-change of a disturbance is easily expressed using the l_∞ norm. Suppose c_r is a rate constraint such that

$$|d(k) - d(k-1)| \leq c_r, \quad \forall k = 1, 2, \ldots .$$

Using the usual backward shift operator $q^{-1}d(k) = d(k-1)$, the rate constraint can be written as

$$|(1-q^{-1})d(k)| \leq c_r, \quad \forall k = 1, 2, \ldots ,$$

which is equivalent to the constraint

$$\left\| \frac{(1-q^{-1})}{c_r} d \right\|_\infty \leq 1. \tag{13}$$

Constraints on magnitude and rate of a scalar signal can be combined by introducing the 2×1 weighting operator

$$W^d = \begin{bmatrix} \dfrac{1}{c_m} \\ \dfrac{1-q^{-1}}{c_r} \end{bmatrix}, \tag{14}$$

where c_m and c_r are values for the respective bounds.

In other situations, entries in W^d might represent "frequency" domain weightings in the form of ratios of polynomials in the shift operator q. Morari and Zafiriou (1989) demonstrated how such weights can be constructed for finite-energy signals. Consider Figure 2, where the signal d' is bounded in magnitude, $\|d'\|_\infty \leq 1$. Signal d is generated by convolving the normalized input d' and a weighting function W. The set of bounded disturbances is given by

$$\mathfrak{D}_d = \{d \in l_\infty : \|d'\|_\infty = \|W^{-1}d\|_\infty \leq 1\}. \tag{15}$$

The weighting function W can be chosen to include specific classes of disturbances. For example, to include unit-step inputs in the disturbance set, let

$$W(q) = \frac{\beta q - 1}{\beta(q-1)} \quad \text{with} \quad \beta > 1.$$

A magnitude bounded signal $d'(k) = \beta^{-k}$, after passing through W, will generate a unit-step input $d(k) = 1$. An input

$$d'(k) = \alpha^{-k} \qquad \alpha > 1, \quad \alpha \neq \beta$$

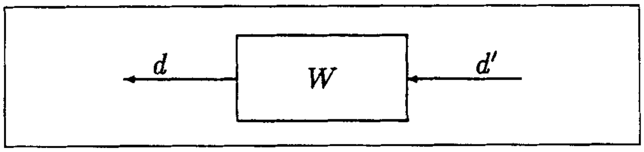

Figure 2. Weighting function for input specification.

satisfies $\|d'\|_\infty \leq 1$, and after passing through W, gives a signal

$$d(k) = \frac{\beta q - 1}{\beta(q-1)} \alpha^{-k},$$

which is a step modified by a lead if $\alpha > \beta$, or by a lag if $\alpha < \beta$. It is clear that judicious choice of W can be used to model a wide variety of disturbances that can occur in process applications.

Sets of admissible control inputs also can be represented by the l_∞ norm formulation. Because manipulated variables are usually subject to constraints in the magnitude and in the rate-of-change, we have found it useful to define the set of feasible control inputs as

$$\mathfrak{D}_u = \{u \in l_\infty^{n_u} : \|W^u u\|_\infty \leq 1\} \tag{16}$$

where

$$W^u = \begin{bmatrix} \dfrac{1}{u_m} \\ \dfrac{1-q^{-1}}{u_r} \end{bmatrix}, \tag{17}$$

and where u_m and u_r are bounds on the magnitude and on the rate of change of the actuator device. As in the case of disturbance modeling, more elaborate weighting functions can be constructed if appropriate for the application.

Performance specification

Performance measures for closed-loop feedback systems include disturbance rejection, set-point tracking, and other dynamical aspects of the output behavior. Quantitative bounds may be imposed on the absolute magnitude of deviations in regulated variables; and on their rates of change, moving averages, and other possible time-domain correlations. In the l_∞ framework presented here, constraints like these can be lumped into a weighting operator that describes regulatory performance.

The regulated output, represented by the vector z in Figure 1, is a composite of many system signals, including process outputs, control action, and setpoint tracking error. Let y_p represent a process output that is a component of z, then a set of acceptable process outputs with respect to a particular class of disturbances is given by

$$\mathfrak{D}_{y_p d} = \left\{ y_p \in l_\infty^{n_y} : \|W^{y_p d} y_p\|_\infty \leq \gamma_d^{\text{perf}} \leq 1 \right\}, \tag{18}$$

where $W^{y_p d}$ is a weighting operator, and where the scalar parameter γ_d^{perf} measures the regulation performance with respect to a process output. A smaller value of γ_d^{perf} implies better performance.

Performance weighting operators are easily constructed for some typical situations. Suppose a_1 and a_2 are bounds imposed on the magnitude and on the rate of change of y_p. $W^{y_p d}$ is then

$$W^{y_p d} = \begin{bmatrix} \dfrac{1}{a_1} \\ \dfrac{1-q^{-1}}{a_2} \end{bmatrix}.$$

If the influence of the disturbance is to be completely suppressed at steady state, an integral action may be imposed by choosing

$$W^{y_p d} = \frac{q^{-1}}{q^{-1}-1}.$$

Setpoint tracking performance can be measured in the same way. Given the error signal $e = r - y_p$, the appropriate setpoint-output set is

$$\mathfrak{D}_{y_p r} = \left\{ y_p \in l_\infty : \|W^{y_p r}(r - y_p)\|_\infty \leq \gamma_r^{\text{perf}} \leq 1 \right\}, \tag{19}$$

where γ_r^{perf} measure setpoint tracking performance.

The output response with respect to sensor noise can be measured as

$$\mathfrak{D}_{y_p n_s} = \left\{ y_p \in l_\infty : \|W^{y_p n_s} y_p\|_\infty \leq \gamma_n^{\text{perf}} \leq 1 \right\}, \tag{20}$$

where γ_n^{perf} is the relevant performance measure.

The preceding discussion focused on scalar components of the feedback control problem. In the multivariable case, off-diagonal elements in the weighting matrices just described can be added to express bounds involving correlated responses among elements in a vector output.

Nominal Control Design Problem

Problem statement

The previous section described methods to specify the class of external inputs that will act on controlled process, and methods to specify and measure classes of acceptable responses. In this section, we pose the nominal control design problem as finding a controller that assures that the output will be in the class of acceptable responses for all inputs in a given class of inputs. The nominal design problem is the task of achieving a desirable mapping between an exogenous input set and an acceptable regulated output set.

A set of exogenous inputs $\mathfrak{D}_w \subseteq l_\infty^{n_w}$ is defined by a finite set of inequalities

$$\mathfrak{D}_w = \{w \in l_\infty^{n_w} : \|W_i^w w\|_\infty \leq 1 \, \forall \, i = 1, \ldots, m\}, \tag{21}$$

where each of the weighting functions is assumed to be stable, causal mapping $W_i^w : l_\infty^{n_w} \to l_\infty$. Also given is a set $\mathcal{Z} \subseteq l_\infty^{n_z}$ of closed-loop regulation objectives of acceptable responses defined by a set of inequality constraints

$$\mathcal{Z} = \{z \in l_\infty^{n_z} : \|W_i^z z\|_\infty \leq 1, \quad i = 1, \ldots, p\}, \tag{22}$$

where the weighting functions are stable, causal mappings $W_i^z : l_\infty^{n_z} \to l_\infty$. The nominal design problem is to find a feedback K such that the closed loop is internally stable and the performance objective is satisfied for all the inputs in the defined set, that is

$$\|W^z z\|_\infty \le 1 \;\forall\, w : \|W^w w\|_\infty \le 1. \tag{23}$$

Solution of the control problem

Polar of the Exogenous Set. The polar of the exogenous input set is used to parameterize the closed loop in terms of convex multipliers. This is a key to the efficient solution of the nominal design problem.

Each constraint on the disturbance set $\mathfrak{D}_w$ can be represented by a convolution of w with a sequence $v_i \in l_1^n$. This is a standard result for any linear time invariant l_∞ bounded operator (Desoer and Vidyasagar, 1975). Therefore the exogenous inputs set also can be written as

$$\mathfrak{D}_w = \{w \in l_\infty^{n_w} : \|v_i * w\|_\infty \le 1 \;\forall\, i = 1,\dots,m\}, \tag{24}$$

where $*$ represents the convolution operation. Each sequence v_i corresponds to the pulse response of the input weighting function W_i^w that appears in Eq. 23. $\mathfrak{D}_w$ is a convex set if $\mathfrak{D}_w$ is nonempty and bounded: there exists a number R such that $\|w\|_\infty \le R \;\forall\, w \in \mathfrak{D}_w$. The polar of $\mathfrak{D}_w$, as adapted from Schrijver (1986), is denoted by $\mathfrak{D}_w^*$ and is defined as those sequences $v \in l_1^{n_v}$ such that $\|v * w\|_\infty \le 1 \;\forall\, w \in \mathfrak{D}_w$. Formally,

$$\mathfrak{D}_w^* = \{v \in l_1^{n_v} : \|v * w\|_\infty \le 1 \forall w \in \mathfrak{D}_w\}. \tag{25}$$

Now consider the convex hull $\hat{\mathfrak{D}}_v$ constructed from the constraints that define the disturbance set

$$\hat{\mathfrak{D}}_v = \Bigg\{ v \in l_1^{n_v} : v = \sum_{i=1}^{m} (\lambda_i^+ - \lambda_i^-) * v_i;\quad \lambda_i^+, \lambda_i^- \in l_1 \forall i;\quad \sum_{i=1}^{m} (\|\lambda_i^+\|_1 + \|\lambda_i^-\|_1) \le 1;\quad \lambda_i^+(k), \lambda_i^-(k) \ge 0 \,\forall\, k \Bigg\}. \tag{26}$$

The following theorem (Keenan and Kantor, 1989) shows that $\hat{\mathfrak{D}}_v$ and $\mathfrak{D}_w^*$ are, in fact, equivalent.

Theorem 1. If $\hat{\mathfrak{D}}_v$ and $\mathfrak{D}_w^*$ are defined as above, then $\mathfrak{D}_w^* \equiv \hat{\mathfrak{D}}_v$. Equivalently, $\|v * w\|_\infty \le 1 \;\forall\, w \in \mathfrak{D}_w$ if and only if $v \in \hat{\mathfrak{D}}_v$.

This result allows us to characterize the sets of all closed-loop mapping that satisfy certain constraints by using convex multipliers. Using the feedback control $u = -Ky$, the closed-loop mapping from w to z is given by $z = H_{zw}w$; where $H_{zw} = P_{zw} - P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$. If the plant is stabilizable, then through a doubly coprime factorization (Francis, 1987) the set of all stable closed-loop transfer functions can be parameterized as $H_{zw} = T_1 - T_2QT_3$, where T_1, T_2, and T_3 are stable maps. H_{zw} is stable if and only if Q is stable.

Substituting for H_{zw} from Eq. 23, the design objective is to compute Q such that

$$\|W_i^z(T_1 - T_2QT_3)w\|_\infty \le 1 \;\forall\, i = 1, \dots, p \,\forall\, w \in \mathfrak{D}_w. \tag{27}$$

It is evident from the Theorem 1 that we want the pulse response of the weighted closed-loop transfer functions, $W_i^z(T_1 - T_2QT_3)$, to lie in the polar region defined by the disturbance set. Using the parameterization result stated earlier, the nominal control problem is equivalent to the existence of convex multipliers $\lambda_{ij}^\pm$ such that

$$W_i^z(T_1 - T_2QT_3) = \sum_j (\lambda_{ij}^+ - \lambda_{ij}^-) W_j^w \;\forall\, i \tag{28}$$

$$\sum_j \|\lambda_{ij}^+\|_1 + \sum_j \|\lambda_{ij}^-\|_1 \le 1 \tag{29}$$

$$\lambda_{ij}^\pm \ge 0. \tag{30}$$

Here we have replaced the convolution operation by the standard composition of discrete-time operators, according to conventional practice.

Solution by linear programming

An optimal design problem then can be formulated as the task of computing a controller that minimizes an upper bound on $\|W_i^z H_{zw} w\|_\infty$ over the given disturbance set. The parameterization leads to the following linear programming problem:

$$\min_{Q, \lambda_{ij}^\pm} \gamma_{\text{opt}}$$

subject to

$$W_i^z(T_1 - T_2QT_3)_i = \sum_j (\lambda_{ij}^+ - \lambda_{ij}^-) W_j^w \;\forall\, i \tag{31}$$

$$\sum_j \|\lambda_{ij}^+\|_1 + \sum_j \|\lambda_{ij}^-\|_1 \le \gamma_{\text{opt}} \tag{32}$$

$$\lambda_{ij}^\pm \ge 0. \tag{33}$$

Let $\Lambda^\pm$ be the matrix whose elements are the $\lambda_{ij}^\pm$ and let $\Lambda = \Lambda^+ - \Lambda^-$. These equations are then equivalent to

$$\min_{Q, \Lambda^\pm} \gamma_{\text{opt}}$$

subject to

$$W^z(T_1 - T_2QT_3) = (\Lambda^+ - \Lambda^-) W^w \tag{34}$$

$$\|\Lambda^+\|_{i,\infty} + \|\Lambda^-\|_{i,\infty} \le \gamma_{\text{opt}} \tag{35}$$

$$\Lambda_{ij}^\pm \ge 0 \tag{36}$$

The unknowns are γ_{opt}, Q, and the convex parameters $\Lambda^\pm$.

The optimal design problem is an infinite dimensional linear programming problem. A finite-dimensional, suboptimal problem is obtained by choosing a finite time horizon, n_q, for Q. The unknowns are then γ_{opt} and the impulse response coefficients of Q and $\Lambda^\pm$, where

$$Q = \sum_{k=0}^{n_q} Q(k) q^{-k} \tag{37}$$

and

$$\lambda_{ij}^{\pm} = \sum_{k=0}^{n_\lambda} \lambda_{ij}^{\pm}(k) q^{-k}. \tag{38}$$

The solution procedure is to match the coefficients of the forward shift operator Q in Eq. 37. Given a fixed horizon for Q, the horizon n_λ for $\lambda_{ij}^{\pm}$ is chosen so that the right- and lefthand sides have the same order.

Comparison to other l_1 designs

The l_1 problem solved by Dahleh and Pearson (1987, 1988) is the following: Given a set of exogenous signals w that are assumed to be of the form $w = \tilde{W}e$, where e is a bounded input with $\|e\|_\infty \leq 1$, find a stable controller Q that minimizes the maximum amplitude of the regulated output weighted by W^z. That is, find a minimum γ_{opt} so that

$$\|W_z H_{z_w} \tilde{W} e\|_\infty \leq \gamma_{\text{opt}} \tag{39}$$

for all e such that

$$\|e\|_\infty \leq 1. \tag{40}$$

Given a certain W in our formulation, the problems would be equivalent if a weighting function $\tilde{W}$ could be found so that

$$\|W\tilde{W}e\|_\infty \leq 1 \Leftrightarrow \|e\|_\infty \leq 1. \tag{41}$$

It is possible to find such a $\tilde{W}$ only if $W\tilde{W}$ can be made full row rank. This is not generally possible, since W typically has more rows than columns in process-control applications.

Another difference concerns the actual computation. Let the weighted closed-loop transfer function be

$$\Phi = W^{zw} H_{zw} \tilde{W}. \tag{42}$$

Using a Youla parameterization (Francis, 1987), the closed loop can be written as

$$\Phi = T_1 - T_2 Q T_3, \tag{43}$$

where $H \in l_1^{n_z \times n_w}$, $T_1 \in l_1^{n_z \times n_u}$, and $T_2 \in l_1^{n_y \times n_w}$ and Q is a stable parameter in $l_1^{n_u \times n_y}$.

The l_1 problem formulated by the Dahleh and Pearson is then defined as minimizing γ_{opt} such that

$$\gamma_{\text{opt}} = \inf_{Q \text{ stable}} \|T_1 - T_2 Q T_3\|_{i,\infty}. \tag{44}$$

Depending on the dimensions of the different signal spaces, the l_1 control problem can be a one-block problem, also called a good rank problem, if $n_w = n_y$ and $n_z = n_u$. Otherwise it is a multiblock problem, which is also called a "bad rank" problem.

A one-block problem is characterized by a corresponding primal linear program that has a finite number of equality constraints. The multiblock problem, however, is characterized by a primal and dual linear program with an infinite number of variables and constraints. Therefore, only approximate solutions can be generated by truncating the original problem.

Three approximation methods for the l_1 multiblock control problem are reported in the literature. One, known as the *finitely many variables* (FMV) approximation, was developed in Dahleh and Pearson (1988) and McDonald and Pearson (1991). It results from constraining the support of the closed-loop response, Φ, thus providing a suboptimal finitely supported feasible solution to the problem.

In the second approach introduced by Dahleh (1992) and Staffins (1991), known as the *finitely many equations* (FME) approximation, the problem retains only a finite number of constraints in the primal formulation of the problem. The solution generates lower bounds on the optimal solution. A third method introduced by Diaz-Bobillo and Dahleh (1992) is known as the delay augmentation (DA) method. It consists of augmenting the matrices T_2 and T_3 with pure delays such that the augmented problem is one-block, and then applying the techniques developed for one-block problems to the augmented system.

The techniques we presented in the previous sections provide a more direct approach to solution of the l_1 problem. These techniques are not based on interpolation techniques, but rather on the direct parametrization of the closed loop in terms of convex multipliers. This computationally oriented technique does not suffer from the rank restrictions cited earlier.

Regulation with Input Constraints

In this section we demonstrate the application of the ideas presented earlier to the problem of computing a linear feedback regulator that does not violate hard constraints on the control input. For the situation shown in Figure 3, the task is to compute a K that suppresses the influence of a disturbance d on the output y_p. Disturbance d is not measured, but it is from a known class of disturbances. The control input is constrained in magnitude and rate of change.

The exogenous inputs consist of unmeasured disturbances d, that is, $w = d$ that belong to a weighted set

$$\mathfrak{D}_d = \{d \in l_\infty^{n_d} : \|W^d d\|_\infty \leq 1\}, \tag{45}$$

where W^d is chosen according to the guidelines in the earlier sections.

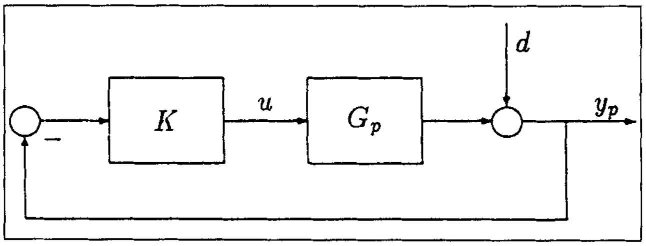

Figure 3. Disturbance regulation with input constraints.

To account for the control input constraints, a weighting operator

$$W^u = \begin{bmatrix} \dfrac{1}{u_m} \\ \dfrac{1-q^{-1}}{u_r} \end{bmatrix}, \tag{46}$$

where u_m is the upper bound on the magnitude of u, and u_r is an upper bound on the magnitude of the rate of change of u.

Given W^d, W^u, and an additional weighting function W^{y_pd}, which measures the process output, the nominal control objectives can be stated as follows:

- Reject unmeasured disturbances d at the output y_p:

$$\|W^{y_pd} y_p\|_\infty \le \gamma_{\text{per}\,f} \le 1 \, \forall d \text{ such that } \|W^d d\|_\infty \le 1.$$

- Achieve the previous objective with the available control action u:

$$\|W^u u\|_\infty \le 1 \, \forall d \text{ such that } \|W^d d\|_\infty \le 1.$$

The regulated variables comprise the process output y_p and the control input u. The closed-loop transfer function H_{zd} for the feedback system is then

$$\begin{bmatrix} y_p \\ u \end{bmatrix} = \begin{bmatrix} (I - G_p K S_o) G_d \\ -K S_o G_d \end{bmatrix} [d], \tag{47}$$

where $S_o = (1 + G_p K)^{-1}$ is the output sensitivity function.

A first step in solving the nominal design problem is to augment the nominal feedback system with the weighting functions W^{y_pd} and the control inputs constraints W^u. The transfer function of the augmented plant $G_{p,\text{aug}}$ shown in Figure 4 is given by

$$G_{p,\text{aug}} = \left[\begin{array}{c|c} W^{y_pd} G_d & W^{y_pd} G_p \\ 0 & W^u \\ \hline -G_d & -G_p \end{array} \right]. \tag{48}$$

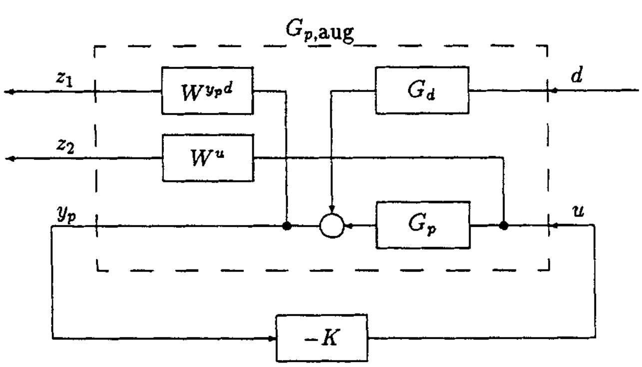

Figure 4. Augmented plant for the disturbance regulation problem.

Given the plant $G_{p,\text{aug}}$, the regulation problem is to find a controller K such that the closed loop is internally stable and the weighted closed loop H_{zd} belongs to the polar of the disturbance set $\mathfrak{D}_d$.

A solution to the regulation can be found by solving the optimization problem

$$\min_{Q,\Lambda} \gamma_{\text{per}\,f} \tag{49}$$

subject to

$$W^{y_pd} H_{zd} = \Lambda W^d \tag{50}$$

$$\|\Lambda\|_{i,\infty} \le \gamma_{\text{per}\,f}. \tag{51}$$

We assume that the plant is stable in the rest of the analysis. The case of an unstable plant is treated in a similar way by using a double coprime factorization. Using the parameterization $Q = KS_o$, the closed-loop transfer function is

$$\begin{bmatrix} y_p \\ u \end{bmatrix} = \begin{bmatrix} (I - G_p Q) G_d \\ -Q G_d \end{bmatrix} [d]. \tag{52}$$

The linear programming problem is to minimize

$$\min_{Q,\Lambda} \gamma_{\text{per}\,f} \tag{53}$$

subject to

$$\begin{bmatrix} W^{y_pd} G_d \\ 0 \end{bmatrix} - \begin{bmatrix} W^{y_pd} G_p \\ W^u \end{bmatrix} Q G_d = \Lambda W^d \tag{54}$$

$$\|\Lambda\|_{i,\infty} \le \gamma_{\text{per}\,f}. \tag{55}$$

It is worth noting that because of the constraints on the manipulative variables, these simple regulation problems are generally multiblock problems since there are more regulated variables than control inputs. Examples of solutions to this problem are given in a later section.

Multiobjective Design for Robust Stability

The control design is further complicated by the presence of model uncertainty. The guaranteed levels of control performance are degraded by uncertainty (Morari and Zafiriou, 1989). Model uncertainties have different sources and they can be described in many different ways, such as bounds on parameters of a linear model, bounds on nonlinearities, and frequency domain bounds.

Here we consider the following robust stabilization problem: Given a plant model and a family of possible true plants, determine conditions for a compensator that stabilizes the nominal plant and stabilizes any plant in the given family. For the purpose of generality, we will assume that the perturbed plant $\tilde{G}_p$ is given by

$$\tilde{G}_p = G_p + W_2 \Delta W_1 \tag{56}$$

where G_p is the nominal plant, and Δ is a strictly causal stable operator such that $\|\Delta\|_{i,\infty} \le 1$. This type of unstructured uncertainty description is quite general (Morari and Zafiriou, 1989). It includes the cases of additive uncertainty ($W_2 = I$), multiplicative input uncertainty ($W_2 = G_p$), and multiplicative output uncertainty ($W_1 = G_p$).

The robust stabilization of plants with unstructured uncertainty typically reduces to the problem of demonstrating stable invertibility of a loop-gain operator for all admissible perturbations (Zames, 1981). Once reduced to this problem of stable invertibility, the small-gain theorem may be used to give a sufficient condition for invertibility, and hence for robust stability. The necessity of the small-gain theorem for the l_∞ bounded input–output operators was proved by Dahleh and Otha (1988). The result states that, given a BIBO linear, shift-invariant causal operator M on l_∞, and a Δ that is a strictly proper operator with $\|\Delta\|_{i,\infty} \le 1$, then the operator $(I + M\Delta)$ has an l_∞-stable inverse with bounded gain for all Δ if and only if $\|M\|_{i,\infty} < 1$.

As an application of this result, suppose that the compensator K stabilizes the nominal plant G_p. Then K stabilizes all the plants $\tilde{G}_p$ if and only if $M = -W_1 K(I + G_p K)^{-1} W_2$ is such that $\|M\|_{i,\infty} < 1$. Using the parameterization described in the last section for stable plants, the expression for M is then

$$M = -W_1 Q W_2. \tag{57}$$

If G_p is not stable, then a double coprime factorization is used to derive an expression of M. In all cases the robust stability equation is linear in Q and can be inserted as a part of the linear programming formulation.

One formulation for solving the robust control problem is to minimize weighted measures of nominal control performance and stability robustness. Two performance indices are introduced:

- $\gamma_{\text{per}\,f}$ to measure nominal performance
- γ_{rs} to measure robust stability.

Each of these should be made small subject to the closed-loop constraints described earlier for the disturbance-rejection problem. The linear programming problem is then

$$\min_{Q,\Lambda_1,\Lambda_2} \alpha_{\text{per}\,f}\gamma_{\text{per}\,f} + \alpha_{rs}\gamma_{rs}, \tag{58}$$

subject to

$$\begin{bmatrix} W^{y_p d} G_d \\ 0 \end{bmatrix} - \begin{bmatrix} W^{y_p d} G_p \\ W^u \end{bmatrix} Q G_d = \Lambda_1 W^d \tag{59}$$

$$-W_1 Q W_2 = \Lambda_2 \tag{60}$$

$$\|\Lambda_1\|_{i,\infty} \le \gamma_{\text{per}\,f} \tag{61}$$

$$\|\Lambda_2\|_{i,\infty} \le \gamma_{rs} \tag{62}$$

$$\gamma_{rs} < 1 \tag{63}$$

$$\alpha_{\text{perf}\,f} + \alpha_{rs} = 1. \tag{64}$$

The parameter γ_{rs} is a measure of stability of robustness of the system. A smaller γ_{rs} is an indication of a more robust stable system. The weighting coefficients $\alpha_{\text{per}\,f}$ and α_{rs} represent relative "costs" associated with nominal performance and robust stability. These weights may be varied to show the tradeoffs between nominal performance and robust stability.

Examples

We demonstrate in this section application of the techniques presented in this article and discuss procedures for computing numerical solutions. Examples 1 and 2, although they do not represent real processes, offer a good way to understand these techniques. In example 3 a chemical process is studied.

Example 1

The first example corresponds to a simple regulation problem where the nominal plant G_p is

$$G_p(z) = \frac{(z-0.3)(z-2.0)}{(z-0.7)(z-0.8)}.$$

The plant is a stable and nonminimum phase. We assume that the disturbance is bounded in magnitude, that is,

$$\|d\|_\infty \le 1, \text{ that is, } W^d = 1,$$

and the magnitude of the output $z = y_p$ is to be minimized

$$\|y_p\|_\infty \le \gamma_d^{\text{per}\,f}, \text{ that is, } W^{y_p d} = 1.$$

We also require steady-state rejection of persistent disturbances. This type of integral action is enforced by choosing

$$W^{y_p d} = \frac{q^{-1}}{1-q^{-1}}.$$

By increasing the horizon n_q of control parameter Q, a sequence of linear programs is generated. This sequence converges for a horizon $n_q = 11$ to a value of $\gamma_d^{\text{per}\,f} = 2.00$. The corresponding controller Q and the multiplier Λ are

$$\begin{aligned} Q = &-1.00 + 1.2z^{-1} - 0.20z^{-2} - 0.06z^{-3} \\ &-0.018z^{-4} - 0.0054z^{-5} - 0.00162z^{-6} - 0.00049z^{-7} \\ &-0.00015z^{-8} - 0.00004z^{-9} - 0.00001z^{-10} \\ \Lambda &= 2.00z^{-1}. \end{aligned}$$

The preceding constraint on d includes disturbances that can vary between 1 and -1 at each sample instant. A more realistic disturbance set is obtained by introducing a bound on its rate of change. As an example, consider

$$W^d = \begin{bmatrix} 1 \\ \dfrac{1-q^{-1}}{0.4} \end{bmatrix}.$$

The weighting function W^d is then nonsquare, and as a result, this problem cannot be treated using the original techniques developed by Dahleh and Pearson. Setting

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

the problem is solved as a medium-sized linear program. The linear program converges for a horizon $n_q = 11$ to $\gamma_d^{\text{per } f} = 1.600$. The controllers Q and Λ are

$$\begin{aligned} Q = & -3.00 + 5.60z^{-1} - 3.0z^{-2} + 0.22z^{-3} + 0.066z^{-4} \\ & + 0.0198z^{-5} + 0.00594z^{-6} + 0.00178z^{-7} + 0.00053z^{-8} \\ & + 0.00016z^{-9} + 0.00005z^{-10} + 0.00001z^{-11} \end{aligned}$$

$$\Lambda = [0 \quad 1.60z^{-1}].$$

The optimal design in this case leads to better performances (γ_{opt} smaller). The addition of a rate constraint to the disturbance set specification has the effect of tightening the set, and hence the optimal design should lead to better performances.

Example 2: a bad rank MIMO problem

In this maximum input/maximum output (MIMO) regulation problem with input constraints, reported by Dahleh and Pearson (1987), we aim to show that the techniques developed in this article are not restricted by the rank of the control problem and that they work as well for "bad" rank problems.

The open-loop feedback system is given by

$$z = P_{zw}w + P_{zu}u$$
$$y = P_{yw}w + P_{yu}u,$$

where

$$P_{yu} = \begin{bmatrix} \dfrac{2}{z-3} & 0 \\ \dfrac{z-1}{z-3} & \dfrac{1-z}{2z} \end{bmatrix}, \qquad P_{yw} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$P_{zu} = \begin{bmatrix} \dfrac{-2}{z-3} & 0 \end{bmatrix}, \qquad P_{zw} = [1 \quad 1].$$

The exogenous inputs are assumed to be bounded in magnitude by 1 and in the rate of change by 0.2,

$$W^w = \begin{bmatrix} 1 & 0 \\ \dfrac{1-q^{-1}}{0.2} & 0 \\ 0 & 1 \\ 0 & \dfrac{1-q^{-1}}{0.2} \end{bmatrix}.$$

The manipulated variables are constrained in magnitude,

$$W^u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the regulation objective consists of minimizing the magnitude of the output: $W^{zw} = 1$.

The first step to a solution is to augment the system open-loop state-space presentation to account for the constraints on the manipulated variables. We define a new regulated variable $\hat{z}$ as

$$\hat{z} = \begin{bmatrix} W^{zw}z \\ u \end{bmatrix},$$

which is equivalent to

$$\hat{z} = \begin{bmatrix} W^{zw}P_{zw} \\ 0 \end{bmatrix} w + \begin{bmatrix} W^{zw}P_{zu} \\ I \end{bmatrix} u.$$

The augmented feedback system is then

$$\hat{z} = P_{\hat{z}w}w + P_{\hat{z}u}u$$
$$y = P_{yw}w + P_{yu}u.$$

Performing a double coprime factorization (Francis, 1987), there exists eight stable rational matrices A_1, B_1, A_2, B_2, X_1, Y_1, X_2, and Y_2 such that

$$P_{yu} = A_2^{-1}B_2 = B_1A_1^{-1}$$

$$\begin{bmatrix} A_2 & B_2 \\ -Y_1 & X_1 \end{bmatrix} \begin{bmatrix} X_2 & -B_1 \\ Y_2 & A_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The stabilizing controller is given by

$$K = (Y_2 + A_1Q)(X_2 - B_1Q)^{-1}.$$

A solution to the above equations is given by

$$A_1 = \begin{bmatrix} 0 & \dfrac{z-3}{6z} \\ -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \dfrac{1}{\frac{z-3}{6z}} & -1 \\ & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & -\dfrac{1}{3z} \\ \dfrac{1-z}{2z} & \dfrac{1-z}{6z} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & \dfrac{1-z}{2z} \\ \dfrac{-1}{3z} & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 & -1 \\ 6 & 0 \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} 0 & 0 \\ -9 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 6 \\ -1 & -3 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} 0 & -9 \\ 0 & 0 \end{bmatrix}.$$

The corresponding linear program for the optimal regulation is then

$$\min_{Q,\Lambda} \gamma_w^{\text{per}\,f}$$

$$T_1 - T_2 Q T_3 = \Lambda W^w$$

$$\|\Lambda\|_{i,\infty} \le \gamma_w^{\text{per}\,f}$$

where

$$T_2 = P_{\hat{z}u} A_1, \qquad T_3 = A_2 P_{yw}$$

$$T_1 = P_{\hat{z}u} Y_2 P_{yw} + P_{\hat{z}w}.$$

Substituting in the closed-loop transfer function yields

$$T_1 - T_2 Q T_3 = \begin{bmatrix} 1+\dfrac{3}{z} & 1 \\ -\dfrac{3(z-3)}{2z} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{1}{3z} \\ 0 & -\dfrac{3-z}{6z} \\ -1 & 0 \end{bmatrix} Q \begin{bmatrix} 0 & -1 \\ \dfrac{z-3}{6z} & 0 \end{bmatrix}.$$

We notice that this example is a bad rank problem since the number of regulated variables exceeds the number control inputs. Moreover, the weighting function W^w is not square.

The problem is easily solved using the techniques outlined before. Expanding each Q_{ij} and each Λ_{ij} in terms of its impulse responses

$$Q_{ij} = \sum_{k=0}^{n_q} Q_{ij}(k) q^k, \qquad i,j \in [1,2]$$

and

$$\Lambda_{ij} = \sum_{k=0}^{n_\Lambda} \lambda_{ij}(k) q^k, \qquad i \in [1,3], \quad j \in [1,4].$$

The unknowns are the impulse response coefficients of both Q_{ij} and Λ_{ij}. The linear program converges in one iteration to

$$\gamma_w^{\text{per}\,f} = 2.43653 \qquad \text{and} \qquad Q = \begin{bmatrix} 0 & 0 \\ 1.73494 & 27.00756 \end{bmatrix}.$$

Example 3: Shell benchmark problem

This example is the control of side draw and side end point in the Shell heavy oil fractionator benchmark problem as reported in Prett and García (1988) and Keenan and Kantor (1988). The open-loop discrete-time model for the plant is

$$y = G_p(z) u + G_d(z) d,$$

where

$$G_p(z) = 5.72 z^{-4} \frac{(0.0328 z + 0.0317)}{z - 0.9355}$$

$$G_d(z) = 1.52 z^{-4} \frac{(0.0392 z + 0.1087)}{z - 0.8521}.$$

The disturbance enters the column from the intermediate reflux due to changes in the heat duty requirement from other columns. The plant features uncertain steady-state gains that are parameterized as $K = K_0(1 + W\Delta)$, where $K_0 = 5.72$ is the nominal gain, $W = 0.0996$, and Δ is such that $|\Delta| \le 1$. The measurement device is a pure delay.

The control system must be designed to handle appropriately the following performance criteria:

- Maintain the side draw product end points at specification (0.0 ± 0.005 at steady state).
- Ensure a good closed-loop speed.
- The constraints on the manipulated variable are not to be violated. These constraints include a magnitude bound of 0.5 and a maximum move size limitation of 0.2 per sampling period.
- Ensure stability robustness for the real plant.

In order to do the control design, a disturbance set is to be specified. Disturbances are assumed to be bounded in magnitude and rate:

$$\begin{cases} |d(k)| & \le 1 \\ |d(k) - d(k-1)| & \le 0.5, \end{cases} \tag{65}$$

and hence described by the set

$$\mathfrak{D}_d = \left\{ d \in l_\infty : \left\| \begin{bmatrix} 1 \\ \dfrac{1-q^{-1}}{0.5} \end{bmatrix} d(k) \right\|_\infty \le 1 \right\}. \tag{66}$$

The constraints on the manipulated variables give a set

$$\mathfrak{D}_u = \left\{ u \in l_\infty : \left\| \begin{bmatrix} \dfrac{1}{0.5} \\ \dfrac{1-q^{-1}}{0.2} \end{bmatrix} u(k) \right\|_\infty \le 1 \right\}. \tag{67}$$

Regulation objectives include integral action for the disturbance rejection at y_p. Hence the performance set is given by

$$\mathfrak{D}_{y_p} = \left\{ y_p \in l_\infty : \left\| \left(\frac{q^{-1}}{1-q^{-1}} \right) y_p(k) \right\|_\infty \le 1 \right\}. \tag{68}$$

The nominal design problem is solved first. As shown in Table 1, the sequence of linear programs converges to an optimal solution $\gamma_{\text{per}\,f} = 4.6745$ as the control horizon gets large ($n_q > 40$). This value of $\gamma_{\text{per}\,f}$ indicates that the bounds on the disturbance set have to be attenuated by a factor of $1/\gamma_{\text{per}\,f} = 0.2140$ to avoid violating the constraints on the manipulated variables. An alternative interpretation is that there exist disturbances that, under linear feedback, will saturate the control input by a factor of 4.6745.

Table 1. Convergence of Suboptimal Solutions to Example 3

n_q	$\gamma_{per\ f}$	n_q	$\gamma_{per\ f}$
5	5.11005	30	4.67454
10	4.68435	35	4.67451
15	4.67602	40	4.67450
20	4.67481	45	4.67450
25	4.67461		

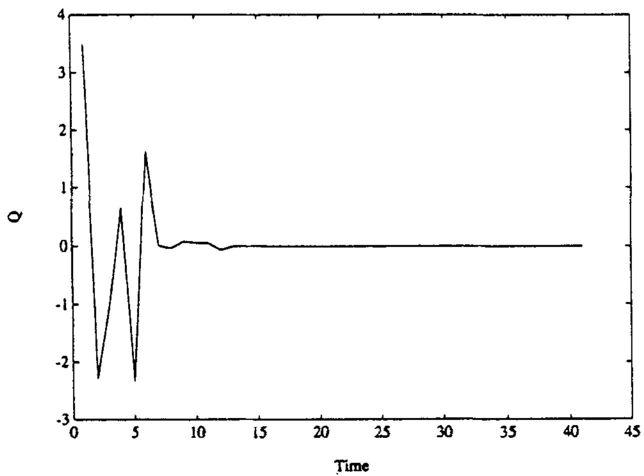

Figure 5. Impulse response of the controller Q for Example 3.

Figure 5 shows the impulse response of the controller Q for a horizon $n_q = 40$. Figure 6 shows the closed-loop response to a step of magnitude $1/\gamma_{per\ f}$. Note that the l_∞ optimal response has a finite impulse response. Figures 7 and 8 show the control input and its rate of change, respectively. The bounds on the manipulative variable are not violated.

Next we solve the nominal problem with an additional constraint for robust stability. The sequence of linear programming problems converges to the optimal solution for a horizon $n_q = 35$. For the case $\alpha_{per\ f} = \alpha_{rs} = 0.5$, the linear program converges to the values $\gamma_{per\ f} = 4.68842$ and $\gamma_{rs} = 0.23210$. This indicates that the maximum bounds on the disturbance set should be $1/\gamma_{per\ f} = 0.2133$. The value of γ_{rs} indicates that the system can still be robustly stable for all $W \leq (W/\gamma_{rs}) = 0.4291$, which allows for a large variation in the gain. It is worth mentioning that because the gain uncertainty is a static (memoryless) uncertainty, the result of the design is conservative.

The tradeoffs between nominal performance and robust stability can be displayed by varying the coefficients $\alpha_{per\ f}$ and α_{rs} and solving the linear program each time. Figure 9 shows the tradeoff between the robust stability and nominal performance. As the nominal performance improves ($\gamma_{per\ f}$ small), the feedback suffers a corresponding loss of stability margin (γ_{rs} large).

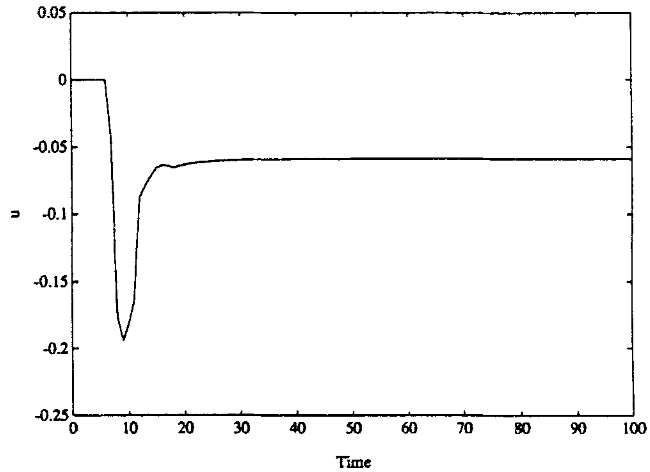

Figure 7. Control input response to a magnitude 0.2140 step disturbance for Example 3.

The response satisfies the control constraint $|u(k)| \leq 0.5$.

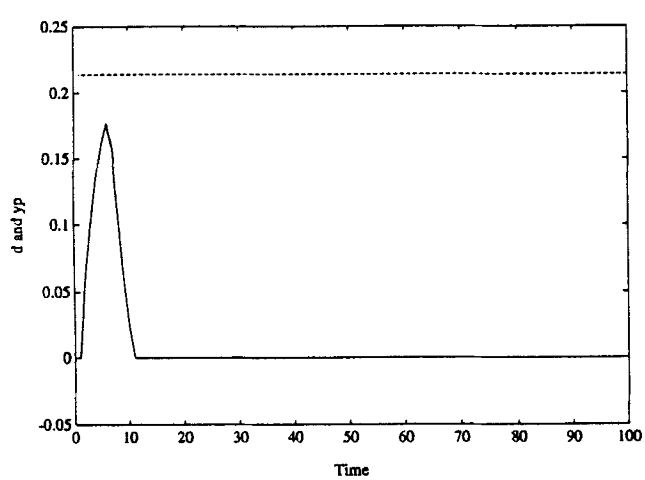

Figure 6. Closed-loop response to a step disturbance of magnitude $1/\gamma$ per $f = 0.2140$.

Legend: distance—dashed line; output—solid line.

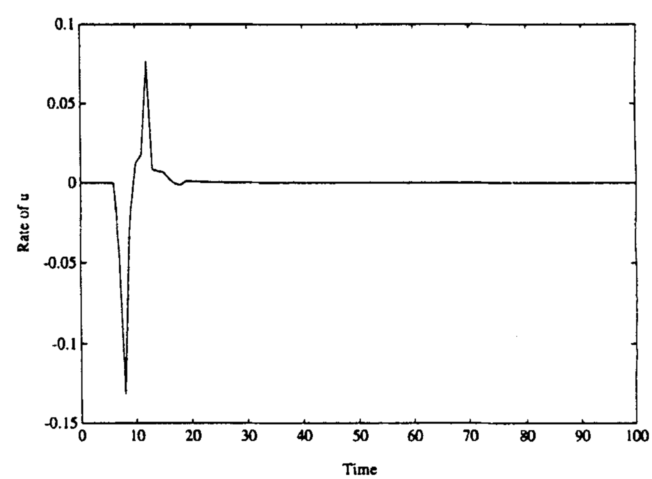

Figure 8. Rate of change of the control input response to a magnitude 0.2140 step disturbance for Example 3.

The response satisfies the control constraint $|u(k) - u(k-1)| \leq 0.2$.

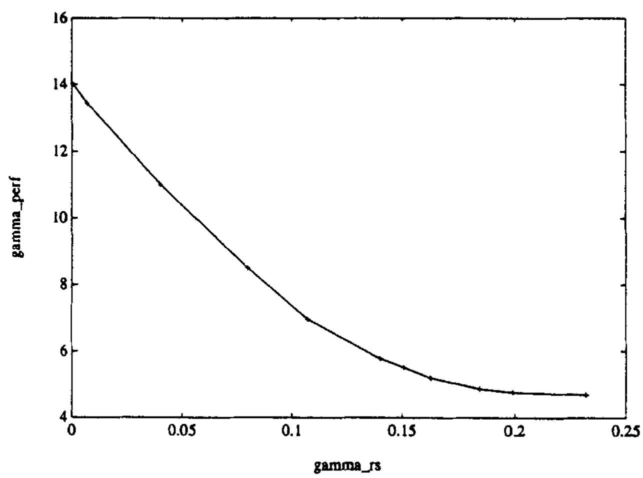

Figure 9. Tradeoff between nominal performance and robust stability for Example 3.

Conclusions

This article presents a novel computational procedure for the design of optimal linear feedback controllers. The controller is computed based on a consideration of the class of expected disturbances and a nominal process model. The infinity norm formulation appears to offer a flexible and intuitive approach to practical constraint-handling control design. The input–output specifications are given in time domain, and the impulse-response coefficients of the optimal compensator are obtained by solving a medium-sized linear program. Since the techniques developed in this article are not based on interpolation methods, they do not suffer from the restrictions associated with the rank of the control problem.

The order of the resulting digital compensator is generally high. Moreover, due to the linear aspect of the design, the compensator action is designed to avoid constraints and saturation rather than "ride" them. These difficulties are shared by most modern approaches to linear control system design. We expect that model reduction would be important to any practical application of the results we've described, and that our techniques may be of interest in the further development of constraint-handling control algorithms.

Acknowledgments

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Notation

$\mathfrak{D}_{y_p d}$ = set of acceptable responses to disturbances
$\mathfrak{D}^*$ = polar of D
$\hat{G}_p$ = true plant
l_1^n = space of n-tuples of elements of l_1. If $x = (x_1, x_2, \ldots, x_n) \in l_1^n$, then the norm is $\|x\|_1 = \sum_{i=1}^n \|x_i\|_1$
l_∞^n = space of n-tuples of elements of l_∞. If $x = (x_1, x_2, \ldots, x_n) \in l_\infty^n$, then the norm is $\|x\|_\infty = \max_i \|x_i\|_\infty$
r = set point
W_1, W_2, W_3 = weighting functions
$\mathfrak{L}_{TV}^{n\times m}$ = space of all bounded linear causal operators from l_∞^n to l_∞^m
$\|\cdot\|$ = the induced operator norm on $\mathfrak{L}_{TV}^{n\times m}$
$\|\cdot\|_{i,\infty}$ = the induced operator norm on $\mathfrak{L}_{TI}^{n\times m}$. If $H \in \mathfrak{L}_{TI}^{n\times m}$, then $\|H\|_{i,\infty} = \max_i \sum_{j=1}^n \|h_{ij}\|_1$. $\mathfrak{L}_{TI}^{n\times m}$ and $l^{n\times m}$ are therefore isomorphes.

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